A CHARACTERIZATION OF HARMONIC POLYNOMIALS

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In this short note, we prove that entire harmonic functions with polynomial bounds are indeed polynomials. More precisely, we give two proofs of the following result:

Theorem 1. *Suppose u is a harmonic function on $\mathbb{R}^n$, and $s \in [0, \infty)$. If there exists a constant C, and a sequence of radii $\{r_1, r_2, \dots\}$, such that $r_j \to +\infty$ as $j \to \infty$, and*

$$u(x) \leq C|x|^s \quad \text{whenever } |x| = r_j \text{ for some } j = 1, 2, \dots, \tag{1}$$

then u is a polynomial of degree $\leq s$.

Compare with Lemma 5.5 in Chapter 5 of [2]. Two aspects of this theorem makes it a bit more difficult than the usual: First, the inequality (1) is only a one-sided inequality (the left-hand side is just u but not the absolute value of u); second, the inequality (1) is only assumed to hold on a sequence of radii tending to infinity. Below we see how to circumvent these difficulties.

First proof of Theorem 1. The first proof is via kernel representations, similar to that of Lemma 5.5 in Chapter 5 of [2]. Let $B(0, r)$ denote the closed ball centered at 0 of radius r. If v is continuous on $B(0, 1)$ and harmonic in the interior, then the Poisson integral formula says

$$v(x) = \int_{|y|=1} v(y) P_1(x, y) d\sigma(y)$$

where P_1 is the Poisson kernel of the unit ball, and $d\sigma$ is the surface measure on the sphere $\{|y| = 1\}$, normalized so that $\int_{|y|=1} d\sigma(y) = 1$.

We will not need the precise form of P here; all we need is that it is smooth, and that we can differentiate the above formula as many times as we wish, to obtain, for each multiindex α, a bounded kernel $K_\alpha(y)$ such that

$$(\partial_x^\alpha v)(0) = \int_{|y|=1} v(y) K_\alpha(y) d\sigma(y). \tag{2}$$

Indeed, we can take $K_\alpha(y) = \partial_x^\alpha P_1(x, y)|_{x=0}$. We write A_α for a constant for which $|K_\alpha(y)| \leq A_\alpha$ for all $|y| = 1$.

Next, suppose u is as in our theorem. We apply the kernel representation (2) to $v(x) := u(r_j x)$, for $j = 1, 2, \dots$, and any multiindex α with $|\alpha| > s$. Then

$$r_j^{|\alpha|} (\partial_x^\alpha u)(0) = \int_{|y|=1} u(r_j y) K_\alpha(y) d\sigma(y).$$

Similarly, by (2) applied to $v(x) := C r_j^s$ instead, we get

$$0 = \int_{|y|=1} C r_j^s K_\alpha(y) d\sigma(y).$$

Combining the two identities, we get

$$(\partial_x^\alpha u)(0) = r_j^{-|\alpha|} \int_{|y|=1} (u(r_j y) - C r_j^s) K_\alpha(y) d\sigma(y).$$

Putting absolute values on both sides, we see that

$$\begin{aligned} |(\partial_x^\alpha u)(0)| &\leq r_j^{-|\alpha|} \int_{|y|=1} |u(r_j y) - C r_j^s| |K_\alpha(y)| d\sigma(y) \\ &\leq A_\alpha r_j^{-|\alpha|} \int_{|y|=1} C r_j^s - u(r_j y) d\sigma(y), \end{aligned}$$

the last line following since $u(r_j y) \leq C r_j^s$ for all $|y| = 1$. But the mean-value property of harmonic functions give

$$\int_{|y|=1} u(r_j y) d\sigma(y) = u(0).$$

Hence the above gives

$$|(\partial_x^\alpha u)(0)| \leq A_\alpha (C r_j^{s-|\alpha|} - u(0) r_j^{-|\alpha|}).$$

Letting $j \to \infty$, we see that $\partial_x^\alpha u(0) = 0$ for all multiindex α with $|\alpha| > s$. Thus expanding $u(x)$ in power series centered at 0, we see that $u(x)$ is a polynomial in x of degree $\leq s$. $\square$

The above proof relies on integral representation formula for harmonic functions. These may not be available in more general settings (e.g. when one considers solutions to second order uniformly elliptic partial differential equations with variable coefficients). We now give a second proof of the theorem, that relies only on the maximum principle (and Harnack inequalities), and gradient estimates, which are available in a more general context.

We first state the Harnack inequality for positive harmonic functions.

Theorem 2. *(Harnack inequality) Suppose u is a positive harmonic function on $B(x_0, R)$ and continuous up to boundary. Then for any x with $|x - x_0| = r < R$, we have*

$$\frac{1 - r/R}{[1 + (r/R)]^{n-1}} u(x_0) \leq u(x) \leq \frac{1 + (r/R)}{[1 - r/R]^{n-1}} u(x_0).$$

Next comes a weak form of the gradient estimate for harmonic functions.

Theorem 3. *(weak form of gradient estimate) Suppose u is a harmonic function on $B(x_0, R)$ and continuous up to the boundary. Then there exists $C_n > 0$ such that for any $k \in (0, 1)$,*

$$\sup_{B(x_0, kR)} |Du| \leq \frac{C_n}{(1-k)^n R} \sup_{B(x_0, R)} |u|.$$

By iteriating the above estimate, we can also derive the gradient estimate for higher order derivatives.

Corollary 4. *Suppose u is a harmonic function on $B(x_0, R)$ and continuous up to the boundary. Then for any $k \in \mathbb{N}$, there exists $C(n, k) > 0$ such that*

$$\sup_{B(x_0, R/2)} |D^k u| \leq \frac{C(n, k)}{R^k} \sup_{B(x_0, R)} |u|.$$

Proof. As before, it suffices to consider the case $x_0 = 0$, $R = 1$. Noted that for any multiindices $\alpha = (i_1, i_2, ..., i_k)$, $\partial^\alpha u$ is still harmonic. Hence, we may apply the previous gradient estimate to $\partial^\alpha u$.

$$\begin{aligned}\sup_{B(1/2)} |\partial_{i_1}\partial_{i_2}...\partial_{i_k} u| &\leq C_n \sup_{B(3/4)} |\partial_{i_2}...\partial_{i_k} u| \\ &\leq C_n' \sup_{B(7/8)} |\partial_{i_3}...\partial_{i_k} u| \\ &\leq C(n,k) \cdot \sup_{B(1)} |u|.\end{aligned}$$

$\square$

Now we are ready to give another proof of Theorem 1.

Second proof of Theorem 1. Suppose u is as in Theorem 1. Then the maximum principle implies that

$$u(x) \leq C r_j^s \quad \text{whenever } |x| \leq r_j \text{ for some } j = 1, 2, \dots. \tag{3}$$

Hence $C r_j^s - u(x)$ is a non-negative harmonic function on $B(0, r_j)$. The Harnack inequality then implies the existence of a constant $c > 0$, such that

$$C r_j^s - u(x) \leq c(C r_j^s - u(0)) \quad \text{for all } x \in B(0, r_j/2),\ j = 1, 2, \dots,$$

and hence we obtain an upper bound for $-u$ as well: there exists a constant C', such that

$$-u(x) \leq C' r_j^s \quad \text{for all } x \in B(0, r_j/2),\ j = 1, 2, \dots. \tag{4}$$

From (3) and (4), we see that there exists a constant C'', such that

$$|u(x)| \leq C'' r_j^s \quad \text{for all } x \in B(0, r_j/2),\ j = 1, 2, \dots.$$

The gradient estimate then implies the existence of a constant C''', such that

$$|\partial_x^\alpha u(0)| \leq C''' r_j^{s-|\alpha|} \quad \text{for all multiindices } \alpha \text{ and all } j = 1, 2, \dots.$$

In particular, letting $j \to \infty$, we see that $\partial_x^\alpha u(0) = 0$ for all multiindices α with $|\alpha| > s$. We can now finish the proof of the theorem as before. $\square$

Finally we come back and discuss a proof of the Harnack inequality and the weak form of the gradient estimate stated earlier. These can be proved using the maximum principle for harmonic functions only, by constructing appropriate test functions. This is a more robust approach that can be generalized to a wider context. However, the most straight-forward proof of the Harnack inequality and the weak form of the gradient estimate for harmonic functions is via the Poisson integral formula. Let us give the most straight-forward proof below, and refer to Chapter 3.4 of [1] for the more robust proof mentioned above.

Proof of the Harnack inequality. By rescaling, it suffices to consider the case where $R = 1$. By Poisson integral formula,

$$u(x) = \frac{1}{\omega_{n-1}} \int_{\partial B(x_0,1)} \frac{1 - r^2}{|x-y|^n} u(y)\, dy$$

where $\omega_{n-1} = |\partial B(1)|$. By using the following simple inequality, $\frac{1-r}{(1+r)^{n-1}} \leq \frac{1-r^2}{|x-y|^n} \leq \frac{1+r}{(1-r)^{n-1}}$, we have

$$u(x) = \fint_{\partial B(x_0,1)} \frac{1-r^2}{|x-y|^n} u(y)\ dy \leq \fint_{\partial B(x_0,1)} \frac{1+r}{(1-r)^{n-1}} u(y) dy = \frac{1+r}{(1-r)^{n-1}} u(x_0).$$

The other side is completely the same. □

Proof of the weak form of the gradient estimate. By rescaling and translating, we may assume $R = 1$ and $x_0 = 0$. As before, we make use of the Poisson integral formula in which

$$u(x) = \frac{1}{\omega_{n-1}} \int_{\partial B(1)} \frac{1-|x|^2}{|x-y|^n} u(y)\ dy.$$

Differentiating the Kernel $P(x,y) = \frac{1-|x|^2}{|x-y|^n}$, we obtain the following.

$$\frac{\partial}{\partial x^i} \left(\frac{1-|x|^2}{|x-y|^n} \right) = \frac{-2x_i}{|x-y|^n} - \frac{n(x_i - y_i)(1-|x|^2)}{|x-y|^{n+2}}$$

where $x = (x_1, x_2, ..., x_n)$, $y = (y_1, ..., y_n)$. Hence, for $|x| = r$,

$$|\partial_{x_i} P(x,y)| \leq \frac{2r}{(1-r)^n} + \frac{n(1+r)}{(1-r)^n} \leq \frac{C_n}{(1-r)^n}.$$

And thus,

$$|\partial_i u(x)| \leq \frac{C_n}{(1-r)^n} \sup_{B(1)} |u|.$$

Taking maximum and run through all index together with maximum principle, we arrive at the following conclusion.

$$\sup_{B(r)} |Du| \leq \frac{C_n}{(1-r)^n} \sup_{B(1)} |u|.$$

□

References

[1] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition. MR1814364

[2] Elias M. Stein and Rami Shakarchi, *Complex analysis*, Princeton Lectures in Analysis, vol. 2, Princeton University Press, Princeton, NJ, 2003.